Last class: Sum of Squares Alg

11/15/2016

Lemma 2.2 Let $\mathcal{E} = \{P_0(x) = 0, x_1^2 - 1 = 0, \ldots, x_n^2 - 1 = 0\}$ for $P_0 \in \mathbb{R}[x]$.

Either: (1) $\mathcal{E}$ is satisfiable or

(2) $\exists$ degree-$2n$ SOS refutation of $\mathcal{E}$.

$\hookrightarrow$ $-1 = S(x) + \sum_i Q_i(x) \cdot P_i(x)$

SOS

degree $\leq \ell$

Expectations over solutions

Properties: $\mathbb{E}_{x \sim D}[1] = 1$

$\mathbb{E}_{x \sim D}[p^2] \geq 0 \quad \forall\ p \in \mathbb{R}[x]$

Def 2.6 Pseudoexpectation

Let $\mathbb{R}[x]_\ell$ be polys in $\mathbb{R}[x]$ of degree $\le \ell$.

A deg-ℓ "pseudoexpectation" for $\mathbb{R}[x]$ is linear map

$\tilde{E} : \mathbb{R}[x]_\ell \longmapsto \mathbb{R}$ s.t.:

(1*) $\tilde{E}[1] = 1$

(2*) $\tilde{E}[p^2] \geq 0$ if $\deg(p) \le \ell$.

↳ We can label each different pseudoexpectation $\tilde{E}$ by label $\tilde{D}$, ie $\tilde{E}_{\tilde{D}}$. We will call "$\tilde{D}$" a pseudo-distribution.

↳ We say deg-ℓ pseudodistribution $\tilde{D}$ satisfies system $P = \{p_1 = 0, \cdots, p_m = 0\}$ if

$$\boxed{\forall\, Q \in \mathbb{R}[x],\ \forall\, i \in [m],\quad \tilde{E}_{\tilde{D}}[\underbrace{Q(x) \cdot P_i(x)}_{\deg \le \ell}] = 0}$$

Q: How to use?

see reference

Thm 2.7: Let $P = \{P_1 = 0, \ldots, P_m = 0\}$ be polys from $\mathbb{R}[x]$.
Assume P is "explicitly bounded". Then, precisely one of the following holds:

① $\exists$ deg-ℓ SOS proof refuting P, or

② $\exists$ deg-ℓ pseudo-distrib $\tilde{D}$ satisfying P.

One dir of Pf/ Suppose (1) holds. We show (2) does not.
Suppose $\exists$ deg-ℓ refutation of P, i.e. $-1 = R + \sum_i Q_i P_i$.
s.t. $\deg(R, Q_i P_i) \le \ell$.

sos, deg $\le \ell$

Let $\tilde{D}$ be any pseudodistribution.

Then: $-1 - R = \sum_i Q_i P_i \iff \tilde{E}_{\tilde{D}}[-1-R] = \tilde{E}_{\tilde{D}}[\sum_i Q_i P_i] \iff -\tilde{E}[1] - \tilde{E}[R] = \sum_i \tilde{E}[Q_i P_i]$

$\iff -1 - (\text{something} \ge 0) = \sum_i \tilde{E}[Q_i P_i] \implies \exists\ i$ s.t. $\tilde{\mathbb{E}}[Q_i P_i] \ne 0$.

i.e. $\tilde{\mathbb{E}}$ does not satisfy P!

This yields dual algorithm:

Dual Deg-ℓ SoS Algorithm

Input: Polys $p_0, \dots, p_m \in \mathbb{R}[x]$

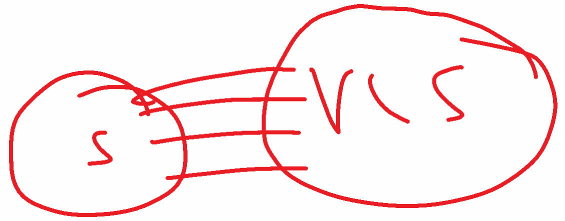
1. output smallest value $\varphi^{(\ell)} \in \mathbb{R}$ s.t. $\exists$ deg-ℓ pseudodistribution satisfying $P = \{P_0 = \varphi^{(\ell)}, P_1 = 0, \dots, P_m = 0\}$.

$\uparrow$ objective

Application to approximating edge expansion

Let $G = (V, E)$ be undirected, d-regular graph.
Let $E(S,T)$ be set of edges from S to T $(S, T \subseteq V)$.
For any vertex set $S \subseteq V$:

$$\phi_G(S) = \frac{|E(S, V \setminus S)| \;\}\ \text{\# edges leaving } S}{d|S| \;\}\ \text{max total \# edges touching } S.}$$



$$\boxed{\phi_G(S) = \frac{|E(S, V\setminus S)|}{d|S|}}$$

Edge expansion of graph (isoperimetric #, cheeger's constant):

$$\phi_G = \min_{\substack{S \subseteq V \\ \text{s.t. } 1 \le |S| \le \frac{|V|}{2}}} \phi_G(S)$$

Let's encode ϕ_G as optimization problem:

Let $x \in \{0,1\}^{|V|}$ be characteristic vector of S, ie $x_i = 1$ iff $v_i \in S$.

(*) Our problem: (for fixed $|S| = k$)

$$\min \frac{1}{\underbrace{dk}_{\text{constant}}} \sum_{(i,j)\in E} (x_i - x_j)^2 \qquad (|E(S, V\setminus S)|) \quad ①$$

$$\text{s.t.} \quad x_i^2 - x_i = 0 \qquad i = 1, \dots, n \qquad (x_i \in \{0,1\}) \quad ②$$

$$\sum_{i=1}^{n} x_i = k \qquad i = 1, \dots, n \qquad (|S| = k) \quad ③$$

Idea: Approximate using deg-ℓ sos!

Let $\phi_G^{(\ell)} = \min_{1 \le k \le \frac{n}{2}} \phi_G^{(\ell,k)}(k)$ be optimal deg-ℓ sos estimate over all $1 \le k \le \frac{n}{2}$.

↳ ϕ_G^{ℓ} can computed in time $n^{O(\ell)}$.

Thm 3.1 $\exists$ constant c s.t. $\forall$ $G(V,E)$,

$$\phi_G \le c\sqrt{\phi_G^{(2)}}.$$

deg-2 SOS estimate of ϕ_G.

Pf / Fix k.

Goal: Given pseudodistribution $\tilde{D}$ over characteristic vectors of size-k sets S s.t.

fixed constant, $= d|S|$, ie denominator in ϕ_G.

$$|\tilde{E}(S, V\setminus S)| = \varphi \cdot dk$$

$\hookrightarrow$ $\tilde{E}: \mathbb{R}^n \mapsto \mathbb{R}$

$\hookrightarrow$ feed in characteristic vector of S $(x_1, \dots, x_n) \in \mathbb{R}^n$,

we want to round $\tilde{D}$ to concrete set $S^* \subseteq V$ s.t.

(a) $|S^*| \le \frac{n}{2}$ (b) $|E(S^*, V\setminus S^*)| \le O(\sqrt{\varphi}) \cdot d|S^*|$.

Assume $k = \frac{n}{2}$.

We know: by (*)

$\tilde{\mathbb{E}}[\sum_i x_i] = \frac{n}{2}$ by ③

$\tilde{\mathbb{E}}[x_i^2 - x_i] = 0$ by ②

$|\tilde{\mathbb{E}}(S, V \setminus S)| = |\tilde{\mathbb{E}}(\sum_{(i,j) \in E} (x_i - x_j)^2)| = \varphi \cdot dk = \varphi d \cdot (\sum_i x_i)$

$\tilde{\mathbb{E}}[1] = 1$

$\tilde{\mathbb{E}}[p^2] \geq 0 \quad \forall p \in \mathbb{R}[x]$ for $\deg(p) \leq 2$.

<u>Alg</u>: Step 1: Choose $y = (y_1, \ldots, y_n)$ from random Gaussian distribution in $\mathbb{R}^n$, with same quadratic moments as $\tilde{D}$, i.e. $\mathbb{E}[y_i] = \tilde{\mathbb{E}}[x_i]$, $\mathbb{E}[y_i y_j] = \tilde{\mathbb{E}}[x_i x_j]$.

Step 2: Output $S^* = \{ i \mid y_i \geq \frac{1}{2} \}$.

Q: How to do step 1?

Recall: n-dim Gaussian dist for $x \in \mathbb{R}^n$:

$$x \sim N_n(\mu, \Sigma) \quad \text{s.t.}$$

① mean vector $\mu = (E[x_1], \cdots, E[x_n])$

② $n \times n$ covariance matrix Σ s.t. $\Sigma(i,j) = E[(X_i - \underbrace{\mu_i}_{E[x_i]})(X_j - \mu_j)]$

↳ symmetric, PSD.

③ For any fixed $a \in \mathbb{R}^n$, then R.V. $y = a^T X$ has univariate Gaussian distribution.

④ Quadratic moments: $E[x_i x_j]$ $\forall\, i,j \in n$.

Idea: For any n-dim distribution over $\mathbb{R}^n$, $\exists$ n-dim Gaussian distribution $\bar{w}$ same quadratic moments.

Pf/ Let D be a distribution over $\mathbb{R}^n$.

Given moments $E[x_i]$ and $E[x_i x_j]$:

① Assume WLOG $E[x_i] = 0 \ \forall i$ (o/w shift variables).

② Consider spectral decomp. of covariance matrix:

$$\Sigma = \sum_i \lambda_i v_i v_i^T,$$ recall $\lambda_i \geq 0$, $\{v_i\}$ o.n. basis for $\mathbb{R}^n$.

③ Define $y \in \mathbb{R}^n$ as $y = \sum_k \sqrt{\lambda_k}\, w_k v_k$

w_k — iid std Gaussian over $\mathbb{R}$.

Claim: ① y is an n-dim Gaussian, ② $E[y_i y_j] = E[x_i x_j]$